Syntactic reduction of predicate tableaux to propositional tableaux

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Abstract: We refine the semantic process which reduces predicate logic to propositional tautologies. We then devise a tableau based proof system mirroring the semantic process by purely syntactic (i.e. programmable) means. We obtain a beautifully symmetric set of theorems given in the figure 1. As a byproduct of the refinement we have a simple tableau method of deciding quasi-tautologies and a proof system without the eigen-variable condition.

1 Introduction

The idea of reducing predicate logic to (propositional) tautologies can be traced back to the Henkin's proof of completeness of predicate calculus [5] but it was first explicitly formulated by Smullyan in the form of his Fundamental theorem [8]. He calls the idea central to predicate logic. A modern and very readable exposition is by Barwise in [1].

The *semantic reduction* of predicate logic to tautologies can be expressed as follows: $\mathbf{T} \models \boldsymbol{A}$ iff the formula $\bigwedge_{1<i\leq n} \boldsymbol{A}_i \rightarrow \boldsymbol{A}$ is a tautology for some $n \geq 0$ such that $\boldsymbol{A}_i \in \mathbf{T} \cup \mathbf{Eq} \cup \mathbf{Q}$. Here $\mathbf{Eq}$ are the *identity* and $\mathbf{Q}$ the *substitution* and *Henkin* axioms.

Smullyan in [8] has investigated the *syntactic*, i.e. constructive, aspects (in this case programmable by primitive recursive functions), of the reduction process.

In this paper we first refine the semantic reduction process by inserting two intermediate stages (*quasi-tautological* and *Henkin reductions*). We then devise a set of tableau expansion rules which permit the exact mirroring of a semantic reduction by the corresponding syntactic reduction. This proves the soundness and completeness of tableaux by purely syntactic means solely from the Fundamental theorem [8]. We do not use Hintikka sets, unifying principles, or systematic tableau processes as it is usually done (see for instance [8, 2]).

We do not claim that our methods, which are *synthetic* (i.e. requiring the *cut condition* and its subsequent elimination), are simpler than or superior to the usual *analytic*, i.e. cut-free, methods. We are interested in exposing, what we think are, beautiful symmetries between the semantic and syntactic constructions. The central concern of the presented paper is the definitions and theorems involved in the equivalences in the figure 1.

The research reported in this paper is a part of a larger project being undertaken by the authors. We are preparing a monograph entitled *Programming by logic and logic by programming* where we investigate logic and arithmetic by syntactic means. These means are programmable by a certain class of subrecursive functions. An extended abstract of the monoghraph was published in [9]. In contrast to the usual approach to logic for computer scientists (see for instance [2]) where proofs (tableaux) are generated automatically, we are concerned (both in the monograph and in the present paper) with the syntactic transformations of tableaux which are already constructed. Such processes are called *finitary characterizations* in [7].

The main reason for the refinement of the reduction process is that it gives us a simple tableau method for deciding quasi-tautologies (this is not reported here) and a proof system without the *eigen-variable* condition (this is reported).

In the section 2 we will sketch the language of predicate logic we will be using. The section 3 covers the refined semantic reduction. Our tableaux are introduced in the section 4 which also discusses the synthetic reduction processes by syntactic means. The section 5 reduces the synthetic condition to the usual analytic one via regularization and cut elimination.

$$\begin{array}{lcl}
\mathbf{T} \models \boldsymbol{A} & \overset{\textbf{(a)}}{\Longleftrightarrow} & \mathbf{T} \vdash \boldsymbol{A} \\
\Updownarrow \quad \text{(Pure formula reduction)} & & \Updownarrow \quad \text{(Cut elimination: theorem 26)} \\
\mathbf{T} \models_h \boldsymbol{A} & \overset{\textbf{(b)}}{\Longleftrightarrow} & \mathbf{T} \vdash^{+} \boldsymbol{A} \\
\Updownarrow \quad \text{(Henkin reduction)} & & \Updownarrow \quad \text{(Intro/elim of q-rules: theorem 20)} \\
\mathbf{T}, \mathbf{Q} \models_i \boldsymbol{A} & & \mathbf{T}, \mathbf{Q} \vdash_i^{+} \boldsymbol{A} \\
\Updownarrow \quad \text{(Quasi-tautological reduction)} & & \Updownarrow \quad \text{(Intro/elim of i-rules: theorem 19)} \\
\mathbf{T}, \mathbf{Q}, \mathbf{Eq} \models_p \boldsymbol{A} & & \mathbf{T}, \mathbf{Q}, \mathbf{Eq} \vdash_p^{+} \boldsymbol{A} \\
\Updownarrow \quad \text{(Tautological reduction)} & & \Updownarrow \quad \text{(Deduction theorem: theorem 18)} \\
\mathit{Taut}(\boldsymbol{As} \rightarrow_s \boldsymbol{A}) \text{ for some } \boldsymbol{As} \subset \mathbf{T} \cup \mathbf{Q} \cup \mathbf{Eq} & \overset{\textbf{(c)}}{\Longleftrightarrow} & \vdash_p^{+} \boldsymbol{As} \rightarrow_s \boldsymbol{A} \text{ for some } \boldsymbol{As} \subset \mathbf{T} \cup \mathbf{Q} \cup \mathbf{Eq}
\end{array}$$

(a): Soundness and completeness of cut-free predicate tableaux

(b): Soundness and completeness of predicate tableaux: theorem 21 **(iii)**

(c): Soundness and completeness of propositional tableaux: theorem 15

Figure 1: Soundness and completeness of tableaux

2 Object language

The *terms* of our object language are formed from infinitely many function symbols of every arity (this includes constants as they have the zero arity) by means of application of functions in the usual way. The *semi-terms* are formed similarly but we permit also *free* variables in them.

Formulas are *semi-formulas* with all variables *bound*. For the construction of semi-formulas from atomic semi-formulas ($\boldsymbol{a} = \boldsymbol{b}$ and $\boldsymbol{p}(\boldsymbol{as})$) we use the full set of propositional connectives $\bot$ (false), $\top$ (true), $\neg$, $\wedge$, $\vee$, $\rightarrow$ (which are right associative and listed in the order of decreasing precedence) and quantifiers $\exists$, $\forall$ in the usual way. *Free* and *bound* occurrences of variables are defined in the usual way. Among the constants we include the set of *special (witnessing) constants* which consists of constants $\boldsymbol{c}_{\exists x A}$ for every existential formula

$\exists \boldsymbol{x} \boldsymbol{A}$. Note that the existential formulas may themselves contain special constants. *Pure* formulas are formulas in which no special constants occur.

We use the syntactic variables $\boldsymbol{f}$, $\boldsymbol{g}$, etc. for function symbols, $\boldsymbol{p}$, $\boldsymbol{q}$, etc. for predicate symbols, $\boldsymbol{a}$, $\boldsymbol{b}$, etc. for semi-terms, $\boldsymbol{A}$ $\boldsymbol{B}$, etc. for semi-formulas, $\boldsymbol{x}$, $\boldsymbol{y}$, etc. for variables and $\boldsymbol{r}$, $\boldsymbol{s}$, etc. for special constants. The reader can go for details to Shoenfield [7] whose notation $\boldsymbol{u}_x[\boldsymbol{a}]$ for substitution in expressions $\boldsymbol{u}$ (i.e. semi-terms and semi-formulas) we also use. We stress that our terms and formulas are *closed*. Semi-terms and semi-formulas are not.

To every syntactic variable we can affix the suffix $\boldsymbol{s}$ to obtain a syntactic variable which ranges over finite (possibly empty) sequences of corresponding objects. The sequence notation $\boldsymbol{a}, \boldsymbol{as}$ stands for a sequence starting with the semi-term $\boldsymbol{a}$ followed by the sequence of semi-terms $\boldsymbol{as}$. $\boldsymbol{As}, \boldsymbol{Bs}$ stands for the concatenation of two sequences of semi-formulas.

We generalize some of the connectives to permit finite sequences. The formula $\boldsymbol{As} \rightarrow_s \boldsymbol{A}$ stands for $\boldsymbol{A}$ if $\boldsymbol{As}$ is empty and for $\boldsymbol{B} \rightarrow \boldsymbol{Bs} \rightarrow_s \boldsymbol{A}$ if the sequence $\boldsymbol{As}$ has the form $\boldsymbol{B}, \boldsymbol{Bs}$. We define $\boldsymbol{As} \vee_s \boldsymbol{A}$ similarly. The sequence $\neg_s \boldsymbol{As}$ is like the sequence $\boldsymbol{As}$ but with every of its formulas negated. The sequence $\boldsymbol{as} =_s \boldsymbol{bs}$ ($\boldsymbol{as} \neq_s \boldsymbol{bs}$) stands for the sequence of equalities (inequalities) between the corresponding elements of the two sequences which are assumed to be of the same length.

We extend the set-theoretic predicates $\in$ and $\subset$ to include finite sequences (eg: $\boldsymbol{A} \in \boldsymbol{As}$, $\boldsymbol{As} \subset \mathbf{T}$) in which case a finite sequence stands for the corresponding finite set formed from the elements of the sequence.

In order to be able to operate on the terms, formulas, tableaux, and finite sequences of those by primitive recursive functions we take these objects to be natural numbers. We leave it to the reader to choose his own favorite *gödelization* of the object language (for details see [4, 9]). From now on we will refer to the codes of finite sequences (which are numbers) as *lists*.

3 Semantic reductions

In order to establish the terminology we start with a quick review of the standard semantic notions for first-order logic. The reader may refer for details to [1]. We then proceed with the investigation of the refined chain of semantic reductions which is done in a detail sufficient to appreciate the symmetries of the figure 1. The chain of semantic reductions in the order discussed below starts in the left bottom corner of the figure and proceeds upwards.

3.1 First-order semantic

A formula $\boldsymbol{A}$ is a *tautology* if it is true only on account of the standard properties of propositional constants and connectives regardless of what truth assignment is given to its *propositional atoms* which are the atomic and quantifier formulas. We write $\mathit{Taut}(\boldsymbol{A})$ if $\boldsymbol{A}$ is a tautology. This is clearly a primitive recursive predicate. One of its definitions may be via the usual method of truth tables.

A *structure* $\mathcal{M}$ is a pair

$$\mathcal{M} = \langle D, I \rangle \tag{1}$$

where D, called the *domain* of $\mathcal{M}$, is a nonempty set and I, called the *interpretation of symbols*, is a mapping which assigns to symbols set-theoretic objects such that

(i) if $\boldsymbol{p}$ is n-ary relation symbol, then $I(\boldsymbol{p}) \subset D^n$ ($D^0 = \{\emptyset\}$);

(ii) if $\boldsymbol{f}$ is n-ary function symbol, then $I(\boldsymbol{f}) : D^n \mapsto D$ ($(D^0 \mapsto D) = D$);

(iii) if $\boldsymbol{r}$ is a special constant, then $I(\boldsymbol{r}) \in D$.

An *assignment in* $\mathcal{M}$ is a function α with domain the set of variables and range a subset of D. We denote by $\alpha(\boldsymbol{x}/d)$ an assignment α_1 which agrees with α everywhere except that $\alpha_1(\boldsymbol{x}) = d$.

For a structure $\mathcal{M}$ from (1) a mapping I from semi-terms and assignments to D is defined in the usual way by induction on semi-terms. We write $\boldsymbol{a}^I[\alpha]$, for the *denotation* in D of the semi-term $\boldsymbol{a}$ under the assignment α. By induction on semi-formulas we define as usual the relation *the semi-formula* $\boldsymbol{A}$ *is true in* $\mathcal{M}$ *for an assignment* α, in writing $\mathcal{M} \models \boldsymbol{A}[\alpha]$. We note that the denotations of terms and the truth values of formulas are fixed regardless of the assignments α. For that reason we abbreviate $\mathcal{M} \models \boldsymbol{A}[\alpha]$ to $\mathcal{M} \models \boldsymbol{A}$ which is read *the formula* $\boldsymbol{A}$ *is true in* $\mathcal{M}$.

For a structure $\mathcal{M}$ the set of formulas $Th(\mathcal{M})$, called the *theory of* $\mathcal{M}$, is $Th(\mathcal{M}) = \{\boldsymbol{A} \mid \mathcal{M} \models \boldsymbol{A}\}$. A structure $\mathcal{M}$ is a *model* of a set of formulas $\mathbf{T}$ if $\mathbf{T} \subset Th(\mathcal{M})$. A formula $\boldsymbol{A}$ is a *logical consequence* of a set of formulas $\mathbf{T}$, written as $\mathbf{T} \models \boldsymbol{A}$, if for every model $\mathcal{M}$ of $\mathbf{T}$ we have $\mathcal{M} \models \boldsymbol{A}$.

We abbreviate $\emptyset \models \boldsymbol{A}$ to $\models \boldsymbol{A}$. This abbreviation is used also with the stronger forms of consequence defined in the following sections.

3.2 Tautological reduction

A *propositional truth set* $\mathbf{S}$ is a subset of formulas such that

(1) $(\neg\boldsymbol{A}) \in \mathbf{S}$ iff $\boldsymbol{A} \notin \mathbf{S}$,

(2) $\top \in \mathbf{S}$ and $\bot \notin \mathbf{S}$,

(3) $(\boldsymbol{A}\wedge\boldsymbol{B}) \in \mathbf{S}$ iff $\boldsymbol{A} \in \mathbf{S}$ and $\boldsymbol{B} \in \mathbf{S}$,

(4) $(\boldsymbol{A}\vee\boldsymbol{B}) \in \mathbf{S}$ iff $\boldsymbol{A} \in \mathbf{S}$ or $\boldsymbol{B} \in \mathbf{S}$,

(5) $(\boldsymbol{A}\rightarrow\boldsymbol{B}) \in \mathbf{S}$ iff $\boldsymbol{A} \notin \mathbf{S}$ or $\boldsymbol{B} \in \mathbf{S}$.

A formula $\boldsymbol{A}$ is a *tautological consequence* of a set of formulas $\mathbf{T}$, written as $\mathbf{T} \models_p \boldsymbol{A}$, if for every propositional truth set $\mathbf{S}$ containing $\mathbf{T}$ also $\boldsymbol{A} \in \mathbf{S}$.

Theorem 1 $Taut(\boldsymbol{A})$ iff $\models_p \boldsymbol{A}$.

Proof: This is straightforward as we use the correspondence between truth tables and propositional truth sets. ∎

Note that a tautology is contained in every propositional truth set.

Theorem 2 (Tautological reduction) We have $\mathbf{T} \models_p \boldsymbol{A}$ iff there is a list of formulas $\boldsymbol{As}$ such that $\boldsymbol{As} \subset \mathbf{T}$ and $Taut(\boldsymbol{As}\rightarrow_s\boldsymbol{A})$.

Proof: This theorem is also known as the *compactness theorem for tautological consequence*. The theorem can be proved either directly or indirectly by using the theorem 4.2 in [1] which is based on the dual notion of propositional satisfiability: A set $\mathbf{S}$ is *propositionally satisfiable* if it can be extended to a propositional truth set. The compactness theorem for propositional satisfiability (the theorem 4.2 in [1]) is: *a set* $\mathbf{S}$ *is propositionally satisfiable iff every list of formulas* $\boldsymbol{As} \subset \mathbf{S}$ *is*. We then use the fact that $\mathbf{T} \models_p \boldsymbol{A}$ iff the set $\mathbf{T} \cup \{\neg\boldsymbol{A}\}$ is propositionally unsatisfiable (and continue with the compactness for satisfiability) iff there is a list $\boldsymbol{As} \subset \mathbf{T}$ such that $\boldsymbol{As} \cup \{\neg\boldsymbol{A}\}$ is propositionally unsatisfiable iff $\boldsymbol{As} \models_p \boldsymbol{A}$ iff $Taut(\boldsymbol{As}\rightarrow_s\boldsymbol{A})$. ∎

3.3 Quasi-tautological reduction

An *identity truth set* $\mathbf{S}$ is a propositional truth set such that

(6) $(\boldsymbol{a}=\boldsymbol{a}) \in \mathbf{S}$,

(7) if $(\boldsymbol{a}=\boldsymbol{b}) \in \mathbf{S}$ then $(\boldsymbol{b}=\boldsymbol{a}) \in \mathbf{S}$,

(8) if $(\boldsymbol{a}=\boldsymbol{b}) \in \mathbf{S}$ and $(\boldsymbol{b}=\boldsymbol{c}) \in \mathbf{S}$ then $(\boldsymbol{a}=\boldsymbol{c}) \in \mathbf{S}$,

(9) if $(\boldsymbol{as}=_s\boldsymbol{bs}) \subset \mathbf{S}$ then $(\boldsymbol{f}(\boldsymbol{as})=\boldsymbol{f}(\boldsymbol{bs})) \in \mathbf{S}$,

(10) if $(\boldsymbol{as}=_s\boldsymbol{bs}) \subset \mathbf{S}$ and $\boldsymbol{p}(\boldsymbol{as}) \in \mathbf{S}$ then $\boldsymbol{p}(\boldsymbol{bs}) \in \mathbf{S}$.

A formula $\boldsymbol{A}$ is a *quasi-tautological consequence* of a set of formulas $\mathbf{T}$, written as $\mathbf{T} \models_i \boldsymbol{A}$, if for every identity truth set $\mathbf{S}$ containing $\mathbf{T}$ also $\boldsymbol{A} \in \mathbf{S}$. The formula $\boldsymbol{A}$ is a *quasi-tautology* if $\models_i \boldsymbol{A}$.

We denote by $\mathbf{Eq}$ the *axioms of identity* which consist of *reflexive*, *symmetric*, *transitive* axioms listed in that order:

$$\boldsymbol{a} = \boldsymbol{a}$$

$$\boldsymbol{a} = \boldsymbol{b} \rightarrow \boldsymbol{b} = \boldsymbol{a}$$

$$\boldsymbol{a} = \boldsymbol{b} \rightarrow \boldsymbol{b} = \boldsymbol{c} \rightarrow \boldsymbol{a} = \boldsymbol{c},$$

and of the axioms of *function* and *predicate substitution* listed in that order:

$$\boldsymbol{as} =_s \boldsymbol{bs} \rightarrow \boldsymbol{f}(\boldsymbol{as}) = \boldsymbol{f}(\boldsymbol{bs})$$

$$\boldsymbol{as} =_s \boldsymbol{bs} \rightarrow \boldsymbol{p}(\boldsymbol{as}) \rightarrow \boldsymbol{p}(\boldsymbol{bs}).$$

The axioms of identity express the desired properties of the identity symbol. The relation between identity and propositional truth sets is captured by the following straightforward lemma:

Lemma 3 $\mathbf{S}$ is an identity truth set iff $\mathbf{S}$ is a propositional truth set containing $\mathbf{Eq}$.

The following theorem reduces propositional logic with identity to propositional logic.

Theorem 4 (Quasi-tautological reduction) We have $\mathbf{T} \models_i \boldsymbol{A}$ iff $\mathbf{T}, \mathbf{Eq} \models_p \boldsymbol{A}$.

Proof: $\mathbf{T} \models_i \boldsymbol{A}$ iff for all identity truth sets $\mathbf{S}$ containing $\mathbf{T}$ also $\boldsymbol{A} \in \mathbf{S}$ iff by the lemma 3 for all propositional truth sets $\mathbf{S}$ containing $\mathbf{T} \cup \mathbf{Eq}$ also $\boldsymbol{A} \in \mathbf{S}$ iff $\mathbf{T}, \mathbf{Eq} \models_p \boldsymbol{A}$. ■

3.4 Henkin reduction

We denote by $\mathbf{H}$ the set of *Henkin witnessing axioms*:

$$\exists \boldsymbol{x} \boldsymbol{A}[\boldsymbol{x}] \rightarrow \boldsymbol{A}[\boldsymbol{c}_{\exists x A}] \tag{2}$$

A model of $\mathbf{H}$ is a *Henkin structure*. A Henkin structure which is also a model of $\mathbf{T}$ is a *Henkin model of* $\mathbf{T}$. A formula $\boldsymbol{A}$ is a *Henkin consequence* of a set of formulas $\mathbf{T}$, written as $\mathbf{T} \models_h \boldsymbol{A}$, if for every Henkin model $\mathcal{M}$ of $\mathbf{T}$ we have $\mathcal{M} \models \boldsymbol{A}$. We clearly have:

Lemma 5 $\mathbf{T} \models_h \boldsymbol{A}$ iff $\mathbf{T}, \mathbf{H} \models \boldsymbol{A}$.

Note that in every Henkin structure the special constant $\boldsymbol{c}_{\exists x A}$ denotes the *witness* such that the formula $\boldsymbol{A}[\boldsymbol{c}_{\exists x A}]$ is true provided that the formula $\exists \boldsymbol{x} \boldsymbol{A}[\boldsymbol{x}]$ is true. In other words, if $\boldsymbol{A}[\boldsymbol{x}]$ can be satisfied at all, it will be satisfied by its witness.

We denote by $\mathbf{Q}$ the set of *quantifier axioms* which consist of $\mathbf{H}$ and of *Henkin counterexample axioms*, *existential*, and *universal substitution axioms* listed in that order:

$$\boldsymbol{A}[\boldsymbol{c}_{\exists x \neg A}] \rightarrow \forall \boldsymbol{x} \boldsymbol{A}[\boldsymbol{x}] \tag{3}$$

$$\boldsymbol{A}[\boldsymbol{a}] \rightarrow \exists \boldsymbol{x} \boldsymbol{A}[\boldsymbol{x}]$$

$$\forall \boldsymbol{x} \boldsymbol{A}[\boldsymbol{x}] \rightarrow \boldsymbol{A}[\boldsymbol{a}].$$

If both the Henkin counterexample axiom (3) and the Henkin witnessing axiom

$$\exists \boldsymbol{x} \neg \boldsymbol{A}[\boldsymbol{x}] \rightarrow \neg \boldsymbol{A}[\boldsymbol{c}_{\exists x \neg A}] \tag{4}$$

hold then the formula $\forall \boldsymbol{x} \boldsymbol{A}[\boldsymbol{x}]$ is true iff its *counterexample* denoted by the special constant $\boldsymbol{c}_{\exists x \neg A}$ is such that the formula $\boldsymbol{A}[\boldsymbol{c}_{\exists x \neg A}]$ is true. The direction $\rightarrow$ is obvious. For the direction $\leftarrow$ we observe that if the formula $\boldsymbol{A}[\boldsymbol{c}_{\exists x \neg A}]$ is true then by (4) we have $\neg \exists \boldsymbol{x} \neg \boldsymbol{A}[\boldsymbol{x}]$ true and thus also $\forall \boldsymbol{x} \boldsymbol{A}[\boldsymbol{x}]$ is true. In other words, if $\boldsymbol{A}[\boldsymbol{x}]$ can be falsified at all, it will be falsified by its counterexample.

Witnessing truth set $\mathbf{S}$ is an identity truth set such that

(11) if $(\exists \boldsymbol{x} \boldsymbol{A}[\boldsymbol{x}]) \in \mathbf{S}$ then $\boldsymbol{A}[\boldsymbol{c}_{\exists x A}] \in \mathbf{S}$,

(12) if $\boldsymbol{A}[\boldsymbol{a}] \in \mathbf{S}$ then $(\exists \boldsymbol{x} \boldsymbol{A}[\boldsymbol{x}]) \in \mathbf{S}$,

(13) if $(\forall \boldsymbol{x} \boldsymbol{A}[\boldsymbol{x}]) \in \mathbf{S}$ then for all terms $\boldsymbol{a}$ we have $\boldsymbol{A}[\boldsymbol{a}] \in \mathbf{S}$,

(14) if $\boldsymbol{A}[\boldsymbol{c}_{\exists x \neg A}] \in \mathbf{S}$ then $(\forall \boldsymbol{x} \boldsymbol{A}[\boldsymbol{x}]) \in \mathbf{S}$.

Witnessing truth sets have a natural interpretation of quantifiers:

Lemma 6 For a witnessing truth set $\mathbf{S}$ we have

(15) $(\exists \boldsymbol{x} \boldsymbol{A}[\boldsymbol{x}]) \in \mathbf{S}$ iff $\boldsymbol{A}[\boldsymbol{a}] \in \mathbf{S}$ for some $\boldsymbol{a}$,

(16) $(\forall \boldsymbol{x} \boldsymbol{A}[\boldsymbol{x}]) \in \mathbf{S}$ iff $\boldsymbol{A}[\boldsymbol{a}] \in \mathbf{S}$ for all $\boldsymbol{a}$.

Proof: **(15)**$\rightarrow$ by **(11)**, **(15)**$\leftarrow$ by **(12)**, and **(16)**$\rightarrow$ by **(13)**. **(16)**$\leftarrow$: if $\boldsymbol{A}[\boldsymbol{a}] \in \mathbf{S}$ for all terms $\boldsymbol{a}$ then also $\boldsymbol{A}[\boldsymbol{c}_{\exists x \neg A}] \in \mathbf{S}$. Hence $(\forall \boldsymbol{x} \boldsymbol{A}[\boldsymbol{x}]) \in \mathbf{S}$ by **(14)**. ■

The relation between witnessing and identity truth sets is captured by the following straightforward lemma:

Lemma 7 $\mathbf{S}$ is a witnessing truth set iff $\mathbf{S}$ is an identity truth set containing $\mathbf{Q}$.

Lemma 8 For a Henkin structure $\mathcal{M}$ the set $Th(\mathcal{M})$ is a witnessing truth set.

Proof: The proof that $Th(\mathcal{M})$ is an identity truth set is straightforward. The properties **(11)**, **(13)**, and **(14)** are shown similarly as the property **(12)** where we assume $\boldsymbol{A}[\boldsymbol{a}] \in Th(\mathcal{M})$, i.e. $\mathcal{M} \models (\boldsymbol{A}[\boldsymbol{a}])[\alpha]$ for any α. Hence also $\mathcal{M} \models (\boldsymbol{A}[\boldsymbol{x}])[\alpha(\boldsymbol{x}/\boldsymbol{a}^I[\alpha])]$ which implies $\mathcal{M} \models (\exists \boldsymbol{x} \boldsymbol{A}[\boldsymbol{x}])[\alpha]$, i.e. $\exists \boldsymbol{x} \boldsymbol{A}[\boldsymbol{x}] \in Th(\mathcal{M})$. ■

Lemma 9 For every witnessing truth set $\mathbf{S}$ there is a Henkin structure with a denumerable domain such that $Th(\mathcal{M}) = \mathbf{S}$.

Proof: We construct a 'Herbrand' structure with the domain a subset of terms in the usual way. See the proof of $(iii) \rightarrow (ii)$ of the Main lemma 4.8 in [1]. ■

Theorem 10 (Henkin consequence) A formula $\boldsymbol{A}$ is a Henkin consequence of a set of formulas $\mathbf{T}$ iff for all witnessing truth sets $\mathbf{S}$ containing $\mathbf{T}$ also $\boldsymbol{A} \in \mathbf{S}$.

Proof: ($\rightarrow$): Assume $\mathbf{T} \models_h \boldsymbol{A}$ and take $\mathbf{S}$ as a witnessing truth set containing $\mathbf{T}$. Construct by the lemma 9 a Henkin structure $\mathcal{M}$ such that $Th(\mathcal{M}) = \mathbf{S}$. Hence $\mathcal{M}$ is a model of $\mathbf{T}$ and so $\boldsymbol{A} \in Th(\mathcal{M}) = \mathbf{S}$.

($\leftarrow$): Assume that for all witnessing truth sets $\mathbf{S}$ containing $\mathbf{T}$ also $\boldsymbol{A} \in \mathbf{S}$ and that $\mathcal{M}$ is a Henkin model of $\mathbf{T}$. By the lemma 8 the set $Th(\mathcal{M})$ is a witnessing truth set containing $\mathbf{T}$. Hence $\boldsymbol{A} \in Th(\mathcal{M})$, i.e. $\mathcal{M} \models \boldsymbol{A}$. ■

Theorem 11 (Henkin reduction) $\mathbf{T} \models_h \boldsymbol{A}$ iff $\mathbf{T}, \mathbf{Q} \models_i \boldsymbol{A}$.

Proof: $\mathbf{T} \models_h \boldsymbol{A}$ iff by the theorem 10 for all witnessing truth sets $\mathbf{S}$ containing $\mathbf{T}$ also $\boldsymbol{A} \in \mathbf{S}$ iff by the lemma 7 for all identity truth sets $\mathbf{S}$ containing $\mathbf{T} \cup \mathbf{Q}$ also $\boldsymbol{A} \in \mathbf{S}$ iff $\mathbf{T}, \mathbf{Q} \models_i \boldsymbol{A}$. ■

3.5 Pure formula reduction

Structures $\mathcal{M}$ and $\mathcal{M}_1$ are *elementarily equivalent on a set of formulas* $\mathbf{T}$ if $Th(\mathcal{M}) \cap \mathbf{T} = Th(\mathcal{M}_1) \cap \mathbf{T}$.

Lemma 12 (Witnessing expansion) To every structure $\mathcal{M}$ there is a Henkin structure $\mathcal{M}_1$ elementarily equivalent with $\mathcal{M}$ on pure formulas.

Proof: The idea is to assign to every special constant $\boldsymbol{c}_{\exists x A}$ a witness from the domain of $\mathcal{M}$ such that $\boldsymbol{A}[\boldsymbol{c}_{\exists x A}]$ is true provided $\exists \boldsymbol{x} \boldsymbol{A}$ is true. For details see the lemma 4.7 in [1]. ■

Theorem 13 (Pure formula reduction) For pure $\mathbf{T}$ and $\boldsymbol{A}$ we have $\mathbf{T} \models \boldsymbol{A}$ iff $\mathbf{T} \models_h \boldsymbol{A}$.

Proof: ($\rightarrow$): Obvious. ($\leftarrow$): Assume $\mathbf{T} \models_h \boldsymbol{A}$ and that $\mathcal{M}$ is a model of $\mathbf{T}$. Construct a Henkin structure $\mathcal{M}_1$ elementarily equivalent on pure formulas with $\mathcal{M}$ by the lemma 12. $\mathcal{M}_1$ is a model of $\mathbf{T}$ and consequently $\mathcal{M}_1 \models \boldsymbol{A}$. Thus also also $\mathcal{M} \models \boldsymbol{A}$. ■

4 Predicate calculus

In this section we are concerned with *syntactic*, as opposed to *semantic*, properties of predicate logic. 'Syntactic' means constructive (finitary) and based on proofs, 'semantic' means based on models.

4.1 Provability

For a decidable predicate $\mathbf{T}$ holding of pure formulas, the predicate

$$Valid_T(\boldsymbol{A}) \text{ iff } \mathbf{T} \models \boldsymbol{A} \text{ and } \boldsymbol{A} \text{ is pure}$$

is a recursively enumerable predicate. This is because from the reduction theorems of the previous section we know that the pure formula $\boldsymbol{A}$ logically follows from $\mathbf{T}$ iff $\mathbf{T} \models_h \boldsymbol{A}$ iff there is a list of formulas $\boldsymbol{As} \subset \mathbf{T} \cup \mathbf{Eq} \cup \mathbf{Q}$ such that $\boldsymbol{As} \rightarrow_s \boldsymbol{A}$ is a tautology. The list $\boldsymbol{As}$, if it exists, is called an *associate of* $\boldsymbol{A}$ *in* $\mathbf{T}$. As the sets $\mathbf{Eq}$ and $\mathbf{Q}$ are decidable by primitive recursive predicates we can see that the predicate $Assoc_T(x, y)$ holding if

> 'x is a list of formulas such that $x \subset \mathbf{T} \cup \mathbf{Eq} \cup \mathbf{Q}$ and $x \rightarrow_s y$ is tautology'

is primitive recursive in $\mathbf{T}$. We have

$$Valid_T(y) \leftrightarrow \exists x \, Assoc_T(x, y).$$

Thus an associate of $\boldsymbol{A}$ in $\mathbf{T}$ can be viewed as a proof of $\boldsymbol{A}$ in $\mathbf{T}$.

Associates are a basis for the notion of provability which can be called *Hilbert systems*. Given a set of formulas $\mathbf{T}$ (not necessarily pure) as *non-logical axioms* we say that the formula $\boldsymbol{A}$ is a *theorem* of the Hilbert system $\mathbf{T}$ if there is a list of formulas, called the *proof* of $\boldsymbol{A}$, such that the last formula of the proof is $\boldsymbol{A}$ and every formula in the proof is either an axiom or else it is the conclusion of the rule modus ponens applied to two preceding formulas. *Axioms* are the tautologies and the formulas from the sets $\mathbf{Eq}$, $\mathbf{Q}$, and $\mathbf{T}$.

Theorem 14 (Soundness and completeness of Hilbert systems) A formula $\boldsymbol{A}$ is a theorem of $\mathbf{T}$ iff $\mathbf{T} \models_h \boldsymbol{A}$.

Proof: ($\rightarrow$): By induction on the length of a proof we show that every theorem has an associate. The empty list is an associate of a tautology. As $\boldsymbol{A} \rightarrow \boldsymbol{A}$ is a tautology, the single element list $\boldsymbol{A}$ is an associate of $\boldsymbol{A} \in \mathbf{T} \cup \mathbf{Eq} \cup \mathbf{Q}$. If by IH the lists $\boldsymbol{As}_1$ and $\boldsymbol{As}_2$ are associates of $\boldsymbol{A}$ and $\boldsymbol{A} \rightarrow \boldsymbol{B}$ respectively, then $\boldsymbol{As}_1, \boldsymbol{As}_2$ is an associate of $\boldsymbol{B}$. By the semantic reduction we have $\mathbf{T} \models_h \boldsymbol{A}$. This proves that Hilbert systems are *sound* in the sense that theorems of $\mathbf{T}$ are Henkin consequences of $\mathbf{T}$.

($\leftarrow$): If $\mathbf{T} \models_h \boldsymbol{A}$ then $\boldsymbol{A}$ has an associate $\boldsymbol{As}$. The formula $\boldsymbol{As} \to_s \boldsymbol{A}$, being a tautology, is a theorem and we get $\boldsymbol{A}$ from it by a series of applications of modus ponens. This proves the *completeness* of Hilbert systems, i.e. that every Henkin consequence of $\mathbf{T}$ is a theorem of $\mathbf{T}$. ■

Our definition of provability in Hilbert systems coincides with the notion of Henkin consequence. By the pure formula reduction Henkin consequence coincides with logical consequence on pure formulas.

The reader familiar with the usual development of formal systems based on modus ponens will note that by formalizing the concept of provability as Henkin consequence we have significantly simplified the axiomatization. We do not need the usual rules of inference of *universal and existential introduction* where, for instance, one proves the formula $\boldsymbol{A} \to \forall \boldsymbol{x}\boldsymbol{B}$ from the premise $\boldsymbol{A} \to \boldsymbol{B}$ provided that the conclusion satisfies the *eigen-variable condition* that the variable $\boldsymbol{x}$ does not freely occur in it. This simplification explains why we have introduced the concept of Henkin consequence.

4.2 Tableaux

The reason why the completeness proof for Hilbert style systems was so easy was that the semantic reductions call for a proof system with the *synthetic consistency property* [8]. This means that the set of theorems must be closed under modus ponens.

We use tableaux dual to the tableaux of Smullyan [8]. Smullyan's tableaux are based on refutation as they show unsatisfiability. Our tableaux demonstrate Henkin consequence. In order to assure the synthetic consistency property we add *cut rules* to our tableaux. We use the syntactic variables $\boldsymbol{X}$, $\boldsymbol{Y}$, etc. to range over tableaux.

A *closed tableau* $\boldsymbol{X}$ *for* $\boldsymbol{As}$ *in* $\mathbf{T}$, in writing $\boldsymbol{X} : \mathbf{T} \vdash^+ \boldsymbol{As}$, establishes $\mathbf{T} \models_h \boldsymbol{As} \vee_s \bot$. It can be visualized as a downward growing dyadic tree whose nodes are formulas. We start the tree with a branch where we write down the formulas from $\boldsymbol{As}$ one after another and then append the tableau $\boldsymbol{X}$ to it. The tableau $\boldsymbol{X}$ is formed from the initial branch $\boldsymbol{As}$ by expanding the branches according to the *rules of expansion* until all branches are *closed*.

The rules of expansions are *unary* or *binary*. A unary rule has the form $\frac{\boldsymbol{As}}{\boldsymbol{B}}$ where we can extend a branch by a formula $\boldsymbol{B}$ provided the formulas of $\boldsymbol{As}$ occur in the branch above. A binary rule has the form $\frac{\boldsymbol{As}}{\boldsymbol{B}\ \boldsymbol{C}}$ where we start two new branches with the formulas $\boldsymbol{B}$ and $\boldsymbol{C}$ provided the formulas of $\boldsymbol{As}$ occur above the fork.

The lists of premises $\boldsymbol{As}$ may be empty. A branch of a tableau can be thought of as a disjunction of its formulas whereas a fork is a conjunction of branches. The leaf at every branch is the *empty tableau* $\bot$.

The tableau expansion rules consist of *propositional*, *identity*, *quantifier*, *axiom*, and *cut* rules. The propositional rules are *negation* rules $\frac{\neg\neg\boldsymbol{B}}{\boldsymbol{B}}$, *disjunction* rules:

$$\frac{\boldsymbol{B} \vee \boldsymbol{C}}{\boldsymbol{B}} \quad \frac{\boldsymbol{B} \vee \boldsymbol{C}}{\boldsymbol{C}} \quad \frac{\boldsymbol{B} \to \boldsymbol{C}}{\neg\boldsymbol{B}}$$

$$\frac{\boldsymbol{B} \to \boldsymbol{C}}{\boldsymbol{C}} \quad \frac{\neg(\boldsymbol{B} \wedge \boldsymbol{C})}{\neg\boldsymbol{B}} \quad \frac{\neg(\boldsymbol{B} \wedge \boldsymbol{C})}{\neg\boldsymbol{C}},$$

and *conjunction* rules:

$$\frac{\boldsymbol{B} \wedge \boldsymbol{C}}{\boldsymbol{B}\ \ \boldsymbol{C}} \quad \frac{\neg(\boldsymbol{B} \to \boldsymbol{C})}{\boldsymbol{B}\ \ \neg\boldsymbol{C}} \quad \frac{\neg(\boldsymbol{B} \vee \boldsymbol{C})}{\neg\boldsymbol{B}\ \ \neg\boldsymbol{C}}.$$

The identity rules correspond to the identity axioms $\mathbf{Eq}$:

$$\frac{}{\boldsymbol{a} \neq \boldsymbol{a}} \quad \frac{\boldsymbol{a} \neq \boldsymbol{b}}{\boldsymbol{b} \neq \boldsymbol{a}} \quad \frac{\boldsymbol{a} \neq \boldsymbol{b}\ \ \boldsymbol{b} \neq \boldsymbol{c}}{\boldsymbol{a} \neq \boldsymbol{c}}$$

$$\frac{\boldsymbol{as} \neq_s \boldsymbol{bs}}{\boldsymbol{f}(\boldsymbol{as}) \neq \boldsymbol{f}(\boldsymbol{bs})} \quad \frac{\boldsymbol{as} \neq_s \boldsymbol{bs}\ \ \neg\boldsymbol{p}(\boldsymbol{as})}{\neg\boldsymbol{p}(\boldsymbol{bs})}.$$

The quantifier rules correspond to the quantifier axioms $\mathbf{Q}$:

$$\frac{\neg\exists \boldsymbol{x}\boldsymbol{A}[\boldsymbol{x}]}{\neg\boldsymbol{A}[\boldsymbol{c}_{\exists x A}]} \quad \frac{\forall \boldsymbol{x}\boldsymbol{A}[\boldsymbol{x}]}{\boldsymbol{A}[\boldsymbol{c}_{\exists x \neg A}]}$$

$$\frac{\exists \boldsymbol{x}\boldsymbol{A}[\boldsymbol{x}]}{\boldsymbol{A}[\boldsymbol{a}]} \quad \frac{\neg\forall \boldsymbol{x}\boldsymbol{A}[\boldsymbol{x}]}{\neg\boldsymbol{A}[\boldsymbol{a}]}.$$

The axiom rules are $\frac{}{\neg\boldsymbol{A}}$ for every $\boldsymbol{A} \in \mathbf{T}$. The cut rules are $\frac{}{\boldsymbol{A}\ \ \neg\boldsymbol{A}}$.

The reader will note that for every unary rule $\frac{\boldsymbol{As}}{\boldsymbol{B}}$ the formula $\boldsymbol{B} \to \boldsymbol{As} \vee_s \bot$ is a tautology (quasi-tautology) [valid in Henkin structures] {propositional consequence of $\mathbf{T}$} if the rule is a propositional (identity) [quantifier] {axiom} rule. For every binary rule $\frac{\boldsymbol{As}}{\boldsymbol{B}\ \boldsymbol{C}}$ the formula $\boldsymbol{B} \wedge \boldsymbol{C} \to \boldsymbol{As} \vee_s \bot$ is a tautology.

A branch of a tableau is *closed* (*p-atomically closed*) [*atomically closed*] if the branch contains $\top$ or $\neg\bot$ or else it contains a pair of formulas (a pair of propositional atoms) [a pair of atomic formulas] $\boldsymbol{B}$ and $\neg\boldsymbol{B}$. A tableau is *closed* (*p-atomically closed*) [*atomically closed*] if every of its branches is *closed* (*p-atomically closed*) [*atomically closed*].

If $\boldsymbol{X} : \mathbf{T} \vdash^{+} \boldsymbol{As}$ and the tableau $\boldsymbol{X}$ is not expanded by quantifier rules we will qualify the provability sign $\vdash^{+}$ by i: $\vdash_i^{+}$ and say that $\boldsymbol{X}$ is an *identity* tableau. The tableau $\boldsymbol{X}$ is a *propositional* tableau if it is an identity tableau without using any identity expansion rules. We will denote this by $\vdash_p^{+}$. In all three forms we will omit the qualifier $+$ if $\boldsymbol{X}$ is a *cut-free* tableau, i.e. a tableau obtained without using the cut rules. It should be clear that in all six forms the predicate $p(x, y) \leftrightarrow x : \mathbf{T} \vdash_{q_1}^{q_2} y$ is a primitive recursive predicate in $\mathbf{T}$. We will write $\mathbf{T} \vdash_{q_1}^{q_2} \mathbf{As}$ in all qualified forms if there is a corresponding tableau $\boldsymbol{X}$ for $\boldsymbol{As}$ in $\mathbf{T}$.

Example 1 The figure 2 shows four tableau proofs. The tableau **(a)** starts with the third formula, the other three tableaux with the second formula. The tableaux **(c)** and **(d)** are in the empty set of axioms. Underneath each tableau is the corresponding provability assertion.

$$
\begin{array}{c}
\neg(\boldsymbol{A} \to \boldsymbol{B}) \\
\boldsymbol{A} \to \boldsymbol{C} \\
\neg\boldsymbol{A} \\
\boldsymbol{C} \\
\begin{array}{cc}
\begin{array}{c} \boldsymbol{A} \\ \bot \end{array} &
\begin{array}{c} \neg\boldsymbol{B} \\ \neg(\boldsymbol{B} \to \boldsymbol{C}) \\
\begin{array}{cc} \begin{array}{c} \boldsymbol{B} \\ \bot \end{array} & \begin{array}{c} \neg\boldsymbol{C} \\ \bot \end{array} \end{array}
\end{array}
\end{array}
\end{array}
$$

$$\{\boldsymbol{B} \to \boldsymbol{C}\} \vdash_p \neg(\boldsymbol{A} \to \boldsymbol{B}), \boldsymbol{A} \to \boldsymbol{C}$$

(a)

$$
\begin{array}{c}
\boldsymbol{a} = \boldsymbol{b} \to \boldsymbol{e} = \boldsymbol{d} \to \boldsymbol{f}(\boldsymbol{a}, \boldsymbol{d}) = \boldsymbol{f}(\boldsymbol{c}, \boldsymbol{e}) \\
\boldsymbol{a} \neq \boldsymbol{b} \\
\boldsymbol{e} = \boldsymbol{d} \to \boldsymbol{f}(\boldsymbol{a}, \boldsymbol{d}) = \boldsymbol{f}(\boldsymbol{c}, \boldsymbol{e}) \\
\boldsymbol{e} \neq \boldsymbol{d} \\
\boldsymbol{f}(\boldsymbol{a}, \boldsymbol{d}) = \boldsymbol{f}(\boldsymbol{c}, \boldsymbol{e}) \\
\boldsymbol{c} \neq \boldsymbol{b} \\
\boldsymbol{b} \neq \boldsymbol{c} \\
\boldsymbol{a} \neq \boldsymbol{c} \\
\boldsymbol{d} \neq \boldsymbol{e} \\
\boldsymbol{f}(\boldsymbol{a}, \boldsymbol{d}) \neq \boldsymbol{f}(\boldsymbol{c}, \boldsymbol{e}) \\
\bot
\end{array}
$$

$$\{\boldsymbol{c} = \boldsymbol{b}\} \vdash_i \boldsymbol{a} = \boldsymbol{b} \to \boldsymbol{e} = \boldsymbol{d} \to \boldsymbol{f}(\boldsymbol{a}, \boldsymbol{d}) = \boldsymbol{f}(\boldsymbol{c}, \boldsymbol{e})$$

(b)

$$
\begin{array}{c}
\exists \boldsymbol{y} \forall \boldsymbol{x} \boldsymbol{A}[\boldsymbol{x}, \boldsymbol{y}] \to \forall \boldsymbol{x} \exists \boldsymbol{y} \boldsymbol{A}[\boldsymbol{x}, \boldsymbol{y}] \\
\neg\exists \boldsymbol{y} \forall \boldsymbol{x} \boldsymbol{A}[\boldsymbol{x}, \boldsymbol{y}] \\
\forall \boldsymbol{x} \exists \boldsymbol{y} \boldsymbol{A}[\boldsymbol{x}, \boldsymbol{y}] \\
\neg\forall \boldsymbol{x} \boldsymbol{A}[\boldsymbol{x}, \boldsymbol{s}] \\
\exists \boldsymbol{y} \boldsymbol{A}[\boldsymbol{r}, \boldsymbol{y}] \\
\neg\boldsymbol{A}[\boldsymbol{r}, \boldsymbol{s}] \\
\boldsymbol{A}[\boldsymbol{r}, \boldsymbol{s}] \\
\bot
\end{array}
$$

$$\boldsymbol{s} = \boldsymbol{c}_{\exists y \forall x A}$$

$$\boldsymbol{r} = \boldsymbol{c}_{\exists x \neg \exists y A}$$

$$\vdash \exists \boldsymbol{y} \forall \boldsymbol{x} \boldsymbol{A}[\boldsymbol{x}, \boldsymbol{y}] \to \forall \boldsymbol{x} \exists \boldsymbol{y} \boldsymbol{A}[\boldsymbol{x}, \boldsymbol{y}]$$

(c)

$$
\begin{array}{c}
\boldsymbol{p}(\boldsymbol{k}) \to \boldsymbol{p}(\boldsymbol{c}_{\exists x\, p(x)}) \\
\neg\boldsymbol{p}(\boldsymbol{k}) \\
\boldsymbol{p}(\boldsymbol{c}_{\exists x\, p(x)}) \\
\begin{array}{cc}
\begin{array}{c} \exists \boldsymbol{x}\, \boldsymbol{p}(\boldsymbol{x}) \\ \boldsymbol{p}(\boldsymbol{k}) \\ \bot \end{array} &
\begin{array}{c} \neg\exists \boldsymbol{x}\, \boldsymbol{p}(\boldsymbol{x}) \\ \neg\boldsymbol{p}(\boldsymbol{c}_{\exists x\, p(x)}) \\ \bot \end{array}
\end{array}
\end{array}
$$

$$\vdash^{+} \boldsymbol{p}(\boldsymbol{k}) \to \boldsymbol{p}(\boldsymbol{c}_{\exists x\, p(x)})$$

(d)

Figure 2: Examples of tableaux

4.3 Syntactic reductions by synthetic means

We will now partially correlate the semantic notions of consequence on the left of the diagram in the figure 1 with the syntactic notions of provability on the right. In this section we will prove the horizontal equivalence **(c)** and the three lower vertical equivalences on the right by purely syntactic means. This will prove the horizontal equivalence **(b)** which is the main result of this section. The topmost vertical equivalence on the right from which the horizontal equivalence **(a)** follows will be proved by syntactic means in the section 5 on analytic (cut-free) tableaux. We use the term 'syntactic' in a strong sense. All equivalences are shown by constructive transformations of tableaux. We are not going to do it explicitly here but it can be shown that all transformations as postulated by the existential theorems and lemmas in this and in the following section can be done by primitive recursive *witnessing* functions. Actually, all witnessing functions fall into the Grzegorczyk class $\mathcal{E}^4$ which is relatively low in the hierarchy of primitive recursive functions.

The reader will note that any tableau $\boldsymbol{X}$ for $\boldsymbol{As}$ in $\mathbf{T}$ remains a tableau for $\boldsymbol{As}_1$ in $\mathbf{T}_1$ when $\boldsymbol{As} \subset \boldsymbol{As}_1$ and $\mathbf{T} \subset \mathbf{T}_1$. This is called *weakening*. We start with the horizontal equivalence **(c)** in the figure 1:

Theorem 15 (Soundness and completeness of axiom-free propositional tableaux) $\mathit{Taut}(\boldsymbol{A})$ iff $\vdash_p \boldsymbol{A}$ iff $\vdash_p^+ \boldsymbol{A}$.

Proof: $\mathit{Taut}(\boldsymbol{A}) \Rightarrow \vdash_p \boldsymbol{A}$ (completeness): We prove by induction on the total number of propositional connectives (excluding the outermost negations) in the formulas of a list $\boldsymbol{As}$ a stronger assertion: $\mathit{Taut}(\boldsymbol{As} \vee_s \bot) \Rightarrow \vdash_p \boldsymbol{As}$. The proof is quite standard and can be obtained by translating into the language of tableaux the proof of the Post's theorem in the section 3.1 of Shoenfield [7].

$\vdash_p \boldsymbol{A} \Rightarrow \vdash_p^+ \boldsymbol{A}$: obvious.

$\vdash_p^+ \boldsymbol{A} \Rightarrow \mathit{Taut}(\boldsymbol{A})$ (soundness): We prove by induction on the structure of the tableau $\boldsymbol{X}$ a more general assertion: $\boldsymbol{X} : \vdash_p^+ \boldsymbol{As} \Rightarrow \mathit{Taut}(\boldsymbol{As} \vee_s \bot)$ using in a straightforward way the fact that the conclusions of every cut and propositional rule tautologically imply the premise. ■

Next we prove two lemmas where we will qualify the provability symbol with qualifiers: $\vdash_{q_1}^{q_2}$. A q_2*-atomic formula* is a propositional atomic formula if q_2 is p or i and an atomic formula otherwise. Similarly, for q_2-atomic closure.

Lemma 16 (Atomic closure) If we have $\boldsymbol{X} : \mathbf{T} \vdash_{q_2}^{q_1} \boldsymbol{As}$ then there is a q_2-atomically closed tableau $\boldsymbol{X}_1$ such that $\boldsymbol{X}_1 : \mathbf{T} \vdash_{q_2}^{q_1} \boldsymbol{As}$.

Proof: If a branch of $\boldsymbol{X}$ closes on a pair $\boldsymbol{B}$ $\neg\boldsymbol{B}$ and $\boldsymbol{B}$ is not a q_2-atomic formula we extend the branch by repeatedly 'opening' both formulas by the corresponding expansion rules until they are q_2-atoms. ■

Lemma 17 (Conjunction inversion) If $\dfrac{\boldsymbol{A}}{\boldsymbol{B}\ \boldsymbol{C}}$ is a conjunction rule and $\boldsymbol{X}$ is a q_2-atomically closed tableau then there are q_2-atomically closed tableaux $\boldsymbol{X}_1 : \mathbf{T} \vdash_{q_2}^{q_1} \boldsymbol{B}, \boldsymbol{As}$ and $\boldsymbol{X}_2 : \mathbf{T} \vdash_{q_2}^{q_1} \boldsymbol{C}, \boldsymbol{As}$.

Proof: If the premise $\boldsymbol{A}$ is of the form $\neg(\boldsymbol{B}\rightarrow\boldsymbol{D})$ where $\boldsymbol{C}$ is $\neg\boldsymbol{D}$ then we construct the tableau $\boldsymbol{X}_1$ by replacing in $\boldsymbol{X}$ every application of the conjunction rule $\dfrac{\neg(\boldsymbol{B} \rightarrow \boldsymbol{D})}{\boldsymbol{B}\ \neg\boldsymbol{D}}$ by its left subtree and omitting both conclusion formulas and the right subtree. The tableau $\boldsymbol{X}_1$ remains q_2-atomically closed and we have $\boldsymbol{X}_1 : \mathbf{T} \vdash_{q_2}^{q_1} \boldsymbol{B}, \neg(\boldsymbol{B}\rightarrow\boldsymbol{D}), \boldsymbol{As}$. We now note that the formula $\neg(\boldsymbol{B} \rightarrow \boldsymbol{D})$ cannot be used in $\boldsymbol{X}_1$ either as a premise of the rule or for the closing of a branch so it can be safely dropped from the assumptions.

The tableau $\boldsymbol{X}_2$ is obtained by a similar operation where we keep only the right subtableau of each use of the above conjunction rule. Proofs of the other kinds of conjunction rules are similar. ■

We are now ready to obtain the syntactic counterparts of the tautological, quasi-tautological, and Henkin reductions which are shown on the right of the figure 1 as the three lower vertical equivalences by the theorems 18, 19, and 20, respectively.

Theorem 18 (Deduction theorem for propositional logic) We have $\mathbf{T} \vdash_p^+ \boldsymbol{A}$ iff $\vdash_p^+ \boldsymbol{As} \rightarrow_s \boldsymbol{A}$ for some $\boldsymbol{As} \subset \mathbf{T}$.

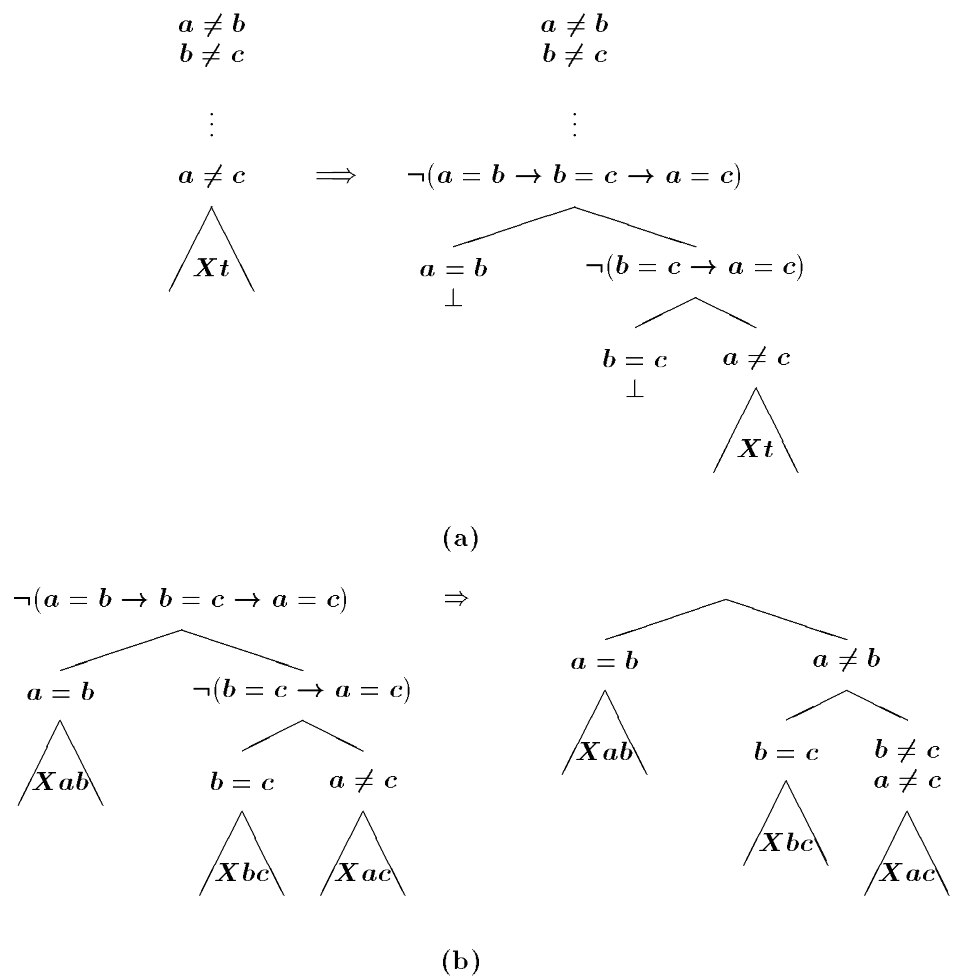


Figure 3: **(a)** Elimination, **(b)** introduction of a transitive rule

Proof: $(\to)$: We collect into the list As all formulas $B \in \mathbf{T}$ where $\neg B$ is a conclusion of an axiom rule in a tableau $X : \mathbf{T} \vdash_p^+ A$. We get $X : As \vdash_p^+ A$. We then delete all conclusions of axiom rules from the tableau X and obtain a new tableau $X_1 : \vdash_p^+ \neg_s As, A$. While the list As is not empty we repeatedly apply the transformation going from $Y : \vdash_p^+ \neg_s Bs, \neg B, C$ to $Y_1 : \vdash_p^+ \neg_s Bs, B\to C$ by starting the tableau Y_1 with the disjunction rule $\frac{B\to C}{\neg B}$ and then affixing Y to it. At the end we obtain a tableau X_2 such that $X_2 : \vdash_p^+ As\to_s A$.

$(\leftarrow)$: For $As \subset \mathbf{T}$ we get from $X : \vdash_p^+ As\to_s A$ to $X_1 : \mathbf{T} \vdash_p^+ A$ as follows. First of all, we may assume that X is p-atomically closed by using the lemma 16 if need be. We then observe that if for a p-atomically closed tableau $Y : \vdash_p^+ (B, Bs)\to_s A$ then also $Y_1 : B \vdash_p^+ Bs \to_s A$ where the p-atomically closed tableau Y_1 is like Y but with all conclusions of the disjunction rule $\frac{B \to Bs \to_s A}{Bs \to_s A}$ deleted. This can be done because every expansion of Y by the disjunction rule $\frac{B\to Bs \to_s A}{\neg B}$ becomes an expansion by the axiom rule $\overline{\neg B}$ in Y_1. Moreover, the p-atomically closed tableau Y cannot close on the formula $(B, Bs) \to_s A$ which is not needed as a premise of any rule either. We then obtain X_1 by repeatedly applying the above transformation starting with X. ∎

Theorem 19 (Introduction and elimination of identity rules) We have $\mathbf{T} \vdash_i^+ A$ iff $\mathbf{T}, \mathbf{Eq} \vdash_p^+ A$.

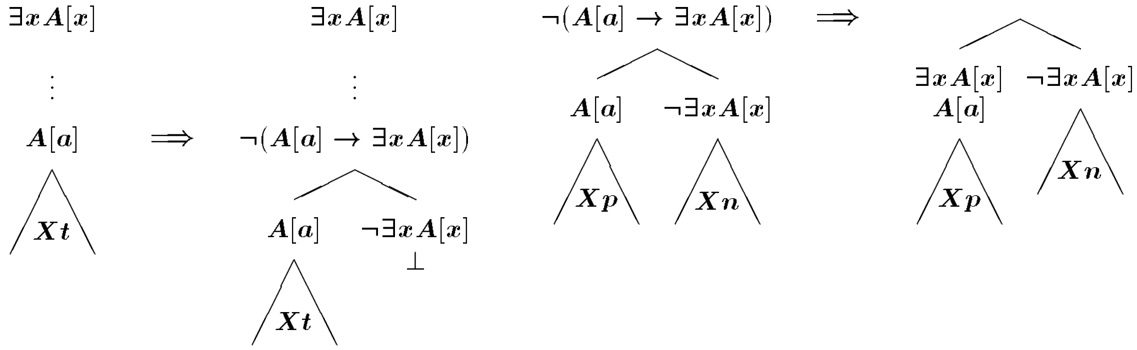
Proof: We use the correspondence between the axioms and rules of identity.

$(\to$:) Given $X : \mathbf{T} \vdash_i^+ A$ we wish to find a tableau $X_1 : \mathbf{T}, \mathbf{Eq} \vdash_p^+ A$. The tableau X_1 is obtained by replacing in X every conclusion of an identity rule by the corresponding identity axiom. The figure 3 **(a)** shows the replacement of a transitive rule. The transformation proceeds in general by repeatedly 'opening' by conjuction rules the inserted identity axiom. The tableau $\boldsymbol{X}_1$ does not rely on any identity rule.

($\leftarrow$:) Given $\boldsymbol{X} : \mathbf{T}, \mathbf{Eq} \vdash_p^+ \boldsymbol{A}$ we wish to find a tableau $\boldsymbol{X}_1 : \mathbf{T} \vdash_i^+ \boldsymbol{A}$. The tableau $\boldsymbol{X}_1$ is obtained by replacing in $\boldsymbol{X}$ every conclusion of an identity axiom by the corresponding identity rule. The figure 3 **(b)** shows this for a transitive axiom. The tableau on the left-hand side is assumed to be in the inverted form. In the general case we do not lose any generality by this as we can first p-atomically close the subtableau under the identity axiom and then repeatedly apply conjunction inversions. The identity axiom and its inverted consequents are then removed and for each formula in its antecedent a cut with the same formula is introduced instead. At the end the rightmost branch will contain the premisses of the corresponding identity rule whose conclusion is introduced by the rule. ■

Theorem 20 (Introduction and elimination of quantifier rules) We have $\mathbf{T} \vdash^+ \boldsymbol{A}$ iff $\mathbf{T}, \mathbf{Q} \vdash_i^+ \boldsymbol{A}$.

Proof: This is similar to the proof of the theorem 19 and uses the correspondence between the quantifier axioms and rules. The figure 4 shows the transformations for an existential substitution rule. ■



(a) **(b)**

Figure 4: **(a)** Elimination, **(b)** introduction of an existential substitution rule

Theorem 21 (Soundness and completeness of tableaux)

(i) $\mathbf{T} \models_p \boldsymbol{A}$ iff $\mathbf{T} \vdash_p^+ \boldsymbol{A}$,

(ii) $\mathbf{T} \models_i \boldsymbol{A}$ iff $\mathbf{T} \vdash_i^+ \boldsymbol{A}$,

(iii) $\mathbf{T} \models_h \boldsymbol{A}$ iff $\mathbf{T} \vdash^+ \boldsymbol{A}$.

Proof: **(i):** $\mathbf{T} \models_p \boldsymbol{A}$ iff by tautological reduction $\mathit{Taut}(\boldsymbol{As} \to_s \boldsymbol{A})$ for some $\boldsymbol{As} \subset \mathbf{T}$ iff by the theorem 15: $\vdash_p^+ \boldsymbol{As} \to_s \boldsymbol{A}$ for some $\boldsymbol{As} \subset \mathbf{T}$ iff by the deduction theorem 18: $\mathbf{T} \vdash_p^+ \boldsymbol{A}$.

(ii): $\mathbf{T} \models_i \boldsymbol{A}$ iff by quasi-tautological reduction $\mathbf{T}, \mathbf{Eq} \models_p \boldsymbol{A}$ iff by **(i)** $\mathbf{T}, \mathbf{Eq} \vdash_p^+ \boldsymbol{A}$ iff by the theorem 19: $\mathbf{T} \vdash_i^+ \boldsymbol{A}$.

(iii): $\mathbf{T} \models_h \boldsymbol{A}$ iff by Henkin reduction $\mathbf{T}, \mathbf{Q} \models_i \boldsymbol{A}$ iff by **(ii)** $\mathbf{T}, \mathbf{Q} \vdash_i^+ \boldsymbol{A}$ iff by the theorem 20: $\mathbf{T} \vdash^+ \boldsymbol{A}$. ■

The completeness result **(iii)** is the best possible. This can be seen for the formula $\boldsymbol{p}(\boldsymbol{k}) \to \boldsymbol{p}(\boldsymbol{c}_{\exists x\, p(x)})$ from the figure 2 **(d)**. Here $\boldsymbol{k}$ is any constant which is not a special constant. By assigning different denotations to $\boldsymbol{k}$ and $\boldsymbol{c}_{\exists x\, p(x)}$ we can easily construct a structure falsifying the formula. Yet, the formula has a tableau proof given in the figure. Hence, logical consequence does not coincide with the $\vdash^+$ provability.

On the other hand, the formula is true in Henkin structures yet it cannot be proved by a cut-free tableau. This is because we cannot get $\boldsymbol{k} \neq \boldsymbol{c}_{\exists x\, p(x)}$ which is necessary to close the tableau by a predicate substitution rule for $\boldsymbol{p}$. Without a cut we cannot use quantifier rules. Hence, Henkin consequence does not coincide with cut-free proofs.

5 Cut elimination

For pure formulas we have a pleasing symmetry of the pure formula reduction with the cut elimination theorem 26. These are the two top vertical equivalences on the left and right sides respectively of the figure 1. The downward implications are trivial. The upward implication on the left, which is semantic, is proved by the witnessing expansion of models of $\mathbf{T}$. The upward implication on the right, which is syntactic, will be proved first by the corresponding operation of *regularization* by the theorem 23 of a proof $\boldsymbol{X} : \mathbf{T} \vdash^{+} \boldsymbol{A}$ and then by a beautiful construction consisting of the cut elimination proper by the theorem 26.

We need some auxiliary concepts first. The *rank* of a formula is the maximum index height of special constants occurring in it. A formula without special constants has rank 0. A formula containing only special constants with index height 1, i.e. $\boldsymbol{c}_{\exists x A}$ where the formula in the index has rank 0, has rank 1, etc. See also [1]. A term $\boldsymbol{a}$ *occurs deeply* in an *expression* $\boldsymbol{u}$ (semi-term, semi-formula, tableau, list or even a set of these) if $\boldsymbol{a}$ occurs directly in $\boldsymbol{u}$ or else if it deeply occurs in the index of a special constant occurring in $\boldsymbol{u}$. A Henkin rule *belongs* to the special constant introduced in its conclusion. Note that on account of the rank argument the special constant cannot occur in the premise of the rule. A list of formulas is *regular* if together with every special constant $\boldsymbol{r}$ deeply occurring in it the list contains both the premise and the conclusion of a Henkin rule belonging to $\boldsymbol{r}$. A tableau $\boldsymbol{X} : \mathbf{T} \vdash^{+} \boldsymbol{As}$ is *regular* if for every subtableau $\boldsymbol{Y}$ for $\boldsymbol{Bs}$ of $\boldsymbol{X}$ the branch $\boldsymbol{Bs}$ (which is above $\boldsymbol{Y}$) is regular. For an expression $\boldsymbol{u}$ we denote by $\boldsymbol{u}_r[[\boldsymbol{a}]]$ the expression obtained from $\boldsymbol{u}$ by the *deep replacement* of the special constant $\boldsymbol{r}$, i.e. by the replacement of every deep occurrence of $\boldsymbol{r}$, in $\boldsymbol{u}$ by the term $\boldsymbol{a}$. A tableau $\boldsymbol{X} : \mathbf{T} \vdash^{+} \boldsymbol{As}$ is *free for the special constant* $\boldsymbol{r}$ if there are no conclusions of Henkin rules belonging to $\boldsymbol{r}$ in $\boldsymbol{X}$.

Lemma 22 (Replacement) If the tableau $\boldsymbol{X} : \mathbf{T} \vdash^{+} \boldsymbol{As}$ is free for the special constant $\boldsymbol{r}$ then $\boldsymbol{X}_r[[\boldsymbol{a}]] : \mathbf{T}_r[[\boldsymbol{a}]] \vdash^{+} \boldsymbol{As}_r[[\boldsymbol{a}]]$ for every term $\boldsymbol{a}$.

Proof: It is easy to see that the replacement does not change the *character* of the proof, i.e. all tableau expansions with the exception of Henkin rules belonging to $\boldsymbol{r}$ remain expansions of the same kind. ■

Theorem 23 (Regularization) To every tableau $\boldsymbol{X} : \mathbf{T} \vdash^{+} \boldsymbol{As}$ with pure $\boldsymbol{As}$ and $\mathbf{T}$ there is a regular tableau $\boldsymbol{X}_1 : \mathbf{T} \vdash^{+} \boldsymbol{As}$.

Proof: We temporarily categorize the topmost deep occurrences of special constants in the above tableau $\boldsymbol{X}$ into three categories: *bound*, *free*, and *critical*. Every special constant deeply occurring in $\boldsymbol{X}$ has a topmost occurrence because $\boldsymbol{As}$ is pure. A topmost deep occurrence of $\boldsymbol{r}$ in a formula $\boldsymbol{A}$ of $\boldsymbol{X}$ is *bound* if the formula is a conclusion of a Henkin rule belonging to $\boldsymbol{r}$, a non-bound occurrence is *free* if the subtableau $\boldsymbol{Y}$ for $\boldsymbol{Bs}$ of $\boldsymbol{X}$ which starts with the formula $\boldsymbol{A}$ is free for $\boldsymbol{r}$, and *critical* otherwise. We note that if the occurrence of $\boldsymbol{r}$ is free we have $\boldsymbol{Y}_r[[\boldsymbol{a}]] : \mathbf{T} \vdash^{+} \boldsymbol{Bs}$ by the lemma 22 because $\mathbf{T}$ is pure and $\boldsymbol{Bs}$ does not contain $\boldsymbol{r}$. This means that we can replace the subtableau $\boldsymbol{Y}$ by a subtableau $\boldsymbol{Y}_r[[\boldsymbol{a}]]$ without changing its character.

Let $\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_i, \ldots, \boldsymbol{r}_n$ be a list of all special constants deeply occurring in $\boldsymbol{X}$ ordered on non-decreasing ranks.

In preparation for an inductive argument on i we assume that no special constant $\boldsymbol{r}_j$ for $i < j$ has a critical occurrence in a tableau $\boldsymbol{Y} : \mathbf{T} \vdash^{+} \boldsymbol{As}$. We will find a tableau $\boldsymbol{Y}_1 : \mathbf{T} \vdash^{+} \boldsymbol{As}$ where also $\boldsymbol{r}_i$ does not have a critical occurrence.

If the special constant $\boldsymbol{r}_i$ has no critical occurrence in $\boldsymbol{Y}$ via a conclusion of a Henkin witnessing rule $\dfrac{\neg\exists \boldsymbol{x} \boldsymbol{B}[\boldsymbol{x}]}{\neg \boldsymbol{B}[\boldsymbol{r}_i]}$ then we set $\boldsymbol{Y}_2$ to $\boldsymbol{Y}$. Otherwise we have the situation on the left of the figure 5 where we have indicated one of the conclusions of the rule. The tableau $\boldsymbol{Y}_2$ shown on the right of the figure is constructed by a cut with $\exists \boldsymbol{x} \boldsymbol{B}[\boldsymbol{x}]$. The left subtree is like $\boldsymbol{Y}$ but with every conclusion $\neg \boldsymbol{B}[\boldsymbol{r}_i]$ replaced by the empty tableau $\perp$. All affected branches remain closed because of the new cut formula. The tableau $\boldsymbol{Y}$ on the right is just weakened. Its new topmost occurrence of $\boldsymbol{r}_i$ is now bound.

If the special constant $\boldsymbol{r}_i$ has no critical occurrence in $\boldsymbol{Y}_2$ via a conclusion of a Henkin counterexample rule we set $\boldsymbol{Y}_1$ to $\boldsymbol{Y}_2$. Otherwise we construct the tableau $\boldsymbol{Y}_1$ similarly as in the previous paragraph.

We note that the special constant $\boldsymbol{r}_i$ has now no critical occurrences in $\boldsymbol{Y}_1$. No of the special constants $\boldsymbol{r}_j$ for $j > i$ can deeply occur in the newly introduced cut formulas and so they remain without critical occurrences in $\boldsymbol{Y}_1$.

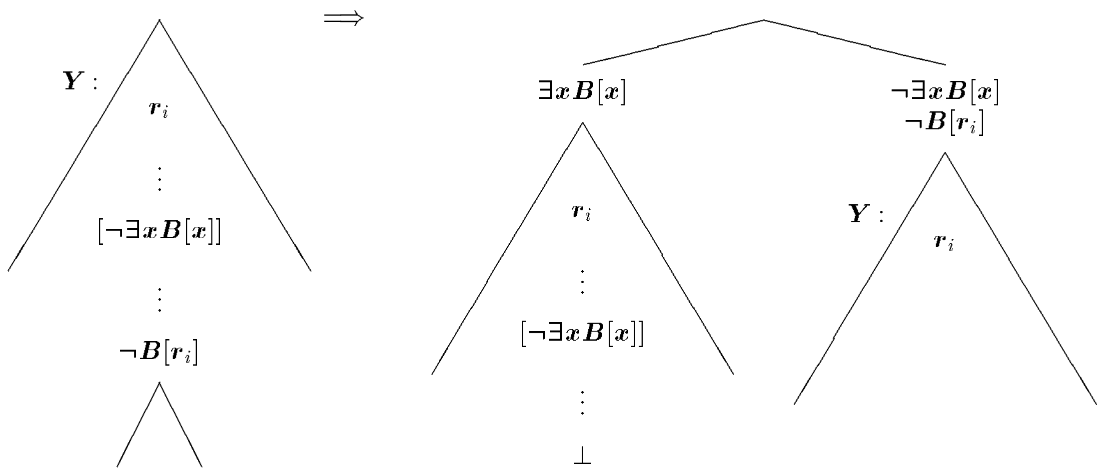


Figure 5: Regularization of tableaux

The tableau $\boldsymbol{X}$ satisfies the above assumptions for $i = n$ and a simple induction shows that there is a tableau $\boldsymbol{X}_2 : \mathbf{T} \vdash^{+} \boldsymbol{As}$ with no critical occurrences of any special constant. The tableau $\boldsymbol{X}_1$ is obtained from $\boldsymbol{X}_2$ by the replacement of all special constants $\boldsymbol{r}_i$ $(i \leq n)$ by a constant $\boldsymbol{k}$ in all subtableaux starting with a free occurrence of $\boldsymbol{r}_i$. All topmost occurrences of special constants are now bound and since $\boldsymbol{As}$ is regular also the tableau $\boldsymbol{X}_1$ is regular. ■

Lemma 24 (Henkin inversion) If $\frac{\boldsymbol{A}}{\boldsymbol{B}}$ is a Henkin (witnessing or counterexample) rule belonging to the special constant $\boldsymbol{r}$ and $\boldsymbol{X} : \mathbf{T} \vdash^{+} \boldsymbol{A}, \boldsymbol{As}$ is a regular atomically closed tableau then there is an atomically closed tableau $\boldsymbol{Y} : \mathbf{T} \vdash^{+} \boldsymbol{B}, \boldsymbol{As}$ free for $\boldsymbol{r}$. Moreover, the tableau constructed from $\boldsymbol{Y}$ by preceding it with the unary conclusion $\boldsymbol{B}$ of $\boldsymbol{A}$ is a regular atomically closed tableau for $\boldsymbol{A}$,$\boldsymbol{As}$ in $\mathbf{T}$.

Proof: If $\boldsymbol{A}$ is $\forall \boldsymbol{x}\boldsymbol{C}[\boldsymbol{x}]$ and $\boldsymbol{B}$ is $\boldsymbol{C}[\boldsymbol{r}]$ where $\boldsymbol{r}$ is $\boldsymbol{c}_{\exists\neg C[x]}$ then we delete all unary conclusions of the form $\neg\neg\boldsymbol{C}[\boldsymbol{r}]$ from $\boldsymbol{X}$ and get the tableau $\boldsymbol{X}_1 : \mathbf{T} \vdash^{+} \neg\neg\boldsymbol{C}[\boldsymbol{r}], \boldsymbol{A}, \boldsymbol{As}$. The tableau $\boldsymbol{X}_1$ does not use the Henkin witnessing rule $\frac{\neg\exists \boldsymbol{x}\neg\boldsymbol{C}[\boldsymbol{x}]}{\neg\neg\boldsymbol{C}[\boldsymbol{r}]}$ also belonging to $\boldsymbol{r}$. We obtain $\boldsymbol{Y}$ from $\boldsymbol{X}_1$ by deleting all unary conclusions $\boldsymbol{C}[\boldsymbol{r}]$ from it. We get $\boldsymbol{Y} : \mathbf{T} \vdash^{+} \boldsymbol{C}[\boldsymbol{r}], \neg\neg\boldsymbol{C}[\boldsymbol{r}], \boldsymbol{A}, \boldsymbol{As}$. We now observe that the formulas $\neg\neg\boldsymbol{C}[\boldsymbol{r}]$ and $\boldsymbol{A}$ when used as premises (of negation and Henkin counterexample rules) have the same conclusions $\boldsymbol{C}[\boldsymbol{r}]$. As these do not occur in the atomically closed tableau $\boldsymbol{Y}$ we may drop both formulas from the assumptions and get $\boldsymbol{Y} : \mathbf{T} \vdash^{+} \boldsymbol{C}[\boldsymbol{r}], \boldsymbol{As}$. The tableau $\boldsymbol{Y}$ is free for $\boldsymbol{r}$. It does not have to be regular because the special constant $\boldsymbol{r}$ which may occur in its assumptions may destroy the regularity of the assumptions. However, the tableau constructed from $\boldsymbol{Y}$ by preceding it with $\boldsymbol{C}[\boldsymbol{r}]$ has the properties required by the lemma.

If the Henkin rule $\frac{\boldsymbol{A}}{\boldsymbol{B}}$ is a witnessing rule then the tableau $\boldsymbol{Y}$ is obtained by a simple deletion of the formula $\boldsymbol{B}$ from the tableau $\boldsymbol{X}$. We leave the proof of this to the reader who is advised to consider that if $\boldsymbol{A}$ is $\neg\exists \boldsymbol{x}\neg\boldsymbol{C}[\boldsymbol{x}]$ then also the Henkin counterexample rule

$\dfrac{\forall x C[x]}{C[r]}$ belongs to r. ■

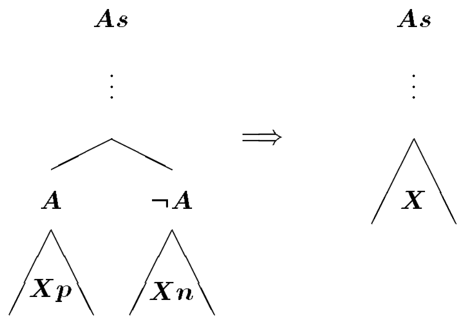
The *size* $\sigma(A)$ of a formula A is the number of propositional connectives and quantifiers occurring in it. The *cut rank* of the conclusion $\overline{A\ ,\ \neg A}$ of a cut rule is $\sigma(\neg A)$. The formula A is the *cut formula*, The *cut rank* $\rho(X)$ of a tableau $X : \mathbf{T} \vdash^+ As$ is the maximal cut rank of all conclusions of cut rules in X. Note that $\rho(X) = 0$ iff X is cut-free.

By the inspection of proofs of the inversion lemmas 17 and 24 we can see that the cut ranks of the tableaux involved are not changed by the inversions and neither is the regularity in the case of conjunction inversion.

The cut elimination theorem was proved originally by G. Gentzen [3] for his sequent calculus. Smullyan [8] translated this argument into the language of tableaux. Our proof is structurally similar to the sequent calculus proof of Schwichtenberg [6]. The following lemma captures the essential step:

Lemma 25 (Reduction lemma) Suppose $Xp : \mathbf{T} \vdash^+ A, As$ and $Xn : \mathbf{T} \vdash^+ \neg A, As$ are regular atomically closed tableaux such that $\rho(Xp) \leq \sigma(A)$ and $\rho(Xn) \leq \sigma(A)$. Then there is a regular atomically closed tableau $X : \mathbf{T} \vdash^+ As$ such that $\rho(X) \leq \sigma(A)$.

Proof: The situation is shown in the following picture:



The tableau on the left is a regular atomically closed tableau for As in $\mathbf{T}$ but with cut rank $\sigma(A) + 1$. In order to get the desired tableau on the right we have to eliminate the cut. We proceed by cases on the form of A.

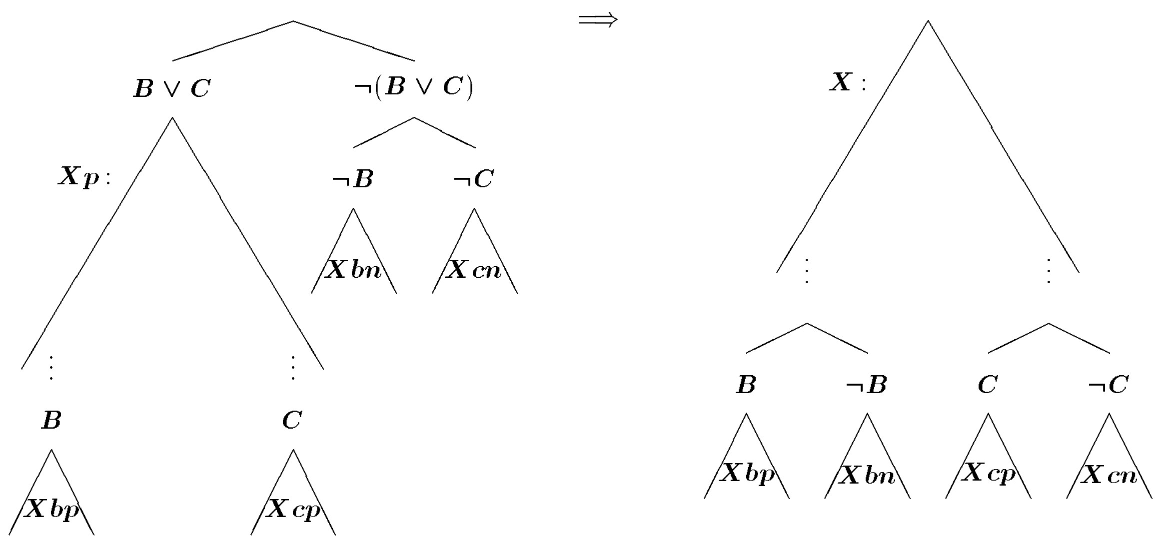


Figure 6: Reduction of a disjunction cut formula

If A is a disjunction $B \lor C$ then the situation is shown on the left of the figure 6 where we have indicated in the left subtableau two typical conclusions of a disjunction rule applied to the cut formula and where the right subtableau is assumed to be already inverted by the conjunction inversion lemma 17. We have $Xbn : \mathbf{T} \vdash^+ \neg B, As$ and $Xcn : \mathbf{T} \vdash^+ \neg C, As$. The tableau $X : \mathbf{T} \vdash^+ As$ shown on the right of the figure is constructed from Xp by inserting into Xp new cuts of rank $\leq \sigma(A)$ instead of all conclusions of disjunction rules for the premise $B \lor C$. The reader will note that the formula $B \lor C$ could have been removed from the assumptions of X as no of its branches can close on the pair $\boldsymbol{B} \vee \boldsymbol{C}$ and $\neg(\boldsymbol{B} \vee \boldsymbol{C})$ and neither is the formula $\boldsymbol{B} \vee \boldsymbol{C}$ a premise of any rule.

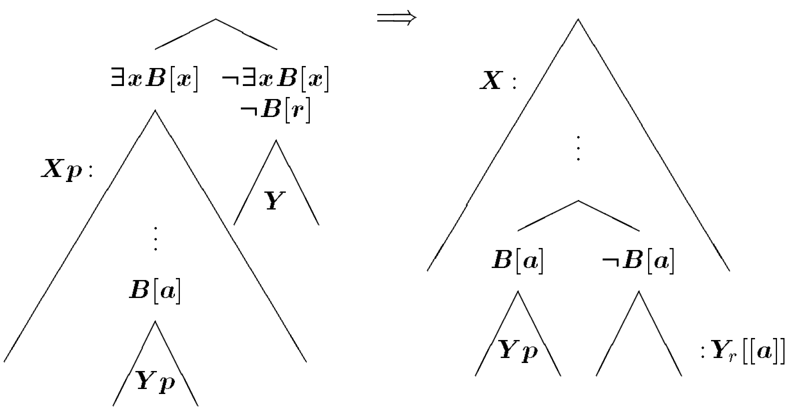


Figure 7: Reduction of an existential cut formula

If $\boldsymbol{A}$ is an existential formula $\exists \boldsymbol{x} \boldsymbol{B}[\boldsymbol{x}]$ then the situation is shown on the left of the figure 7 where we have indicated in the left subtableau one conclusion of an existential substitution rule with the premise $\exists \boldsymbol{x} \boldsymbol{B}[\boldsymbol{x}]$ and where the right subtableau is assumed to be already inverted by the Henkin inversion lemma 24.

There are two subcases to consider. **(a):** The special constant $\boldsymbol{r} = \boldsymbol{c}_{\exists x B}$ deeply occurs in $\boldsymbol{As}$. This is the place in the proof where the regularity assumption is crucial because then $\boldsymbol{As}$ must contain both the premise and a conclusion of a Henkin rule belonging to $\boldsymbol{r}$. If the rule is $\dfrac{\neg\exists \boldsymbol{x} \boldsymbol{B}[\boldsymbol{x}]}{\neg \boldsymbol{B}[\boldsymbol{r}]}$ then we can set $\boldsymbol{X}$ to $\boldsymbol{Y}$ and we are done. If the rule is $\dfrac{\forall \boldsymbol{x} \boldsymbol{C}[\boldsymbol{x}]}{\boldsymbol{C}[\boldsymbol{r}]}$ then $\boldsymbol{B}[\boldsymbol{r}]$ is $\neg \boldsymbol{C}[\boldsymbol{r}]$. Since then $\neg \boldsymbol{B}[\boldsymbol{r}]$ is $\neg\neg \boldsymbol{C}[\boldsymbol{r}]$ it suffices to construct the desired tableau $\boldsymbol{X}$ like the tableau $\boldsymbol{Y}$ but with all unary conclusions $\boldsymbol{C}[\boldsymbol{r}]$ occurring in it removed.

(b): The special constant $\boldsymbol{r}$ does not occur deeply in $\boldsymbol{As}$. The tableau $\boldsymbol{Y} : \mathbf{T} \vdash^{+} \neg \boldsymbol{B}[\boldsymbol{r}], \boldsymbol{As}$ is free for $\boldsymbol{r}$ but it does not have to be regular because the possible occurrence of $\boldsymbol{r}$ in its assumptions without $\neg\exists \boldsymbol{x} \boldsymbol{B}[\boldsymbol{r}]$ may destroy the regularity of the assumptions. We construct the tableau $\boldsymbol{X}$ for $\boldsymbol{As}$ in $\mathbf{T}$ shown on the right of the figure to be like $\boldsymbol{Xp}$ but with the indicated cuts inserted instead of all conclusions $\boldsymbol{B}[\boldsymbol{a}]$ of existential substitution rules with the premise $\exists \boldsymbol{x} \boldsymbol{B}[\boldsymbol{x}]$. The inserted subtableaux $\boldsymbol{Y}_r[[\boldsymbol{a}]] : \mathbf{T} \vdash^{+} \neg \boldsymbol{B}[\boldsymbol{a}], \boldsymbol{Bs}$ where the branches $\boldsymbol{Bs}$ are the formulas above the conclusions are obtained by the replacement lemma 22 and some weakenings. They are regular because the lists $\neg \boldsymbol{B}[\boldsymbol{a}], \boldsymbol{Bs}$ are. The tableau $\boldsymbol{X}$ is obviously atomically closed because the cut formula $\boldsymbol{A}$ cannot be used in it and it is such that $\boldsymbol{\rho}(\boldsymbol{X}) \le \boldsymbol{\sigma}(\boldsymbol{A})$.

If $\boldsymbol{A}$ is an atomic formula then the situation is shown on the left in the figure 8 where we have indicated one of the branches which close on the pair of formulas $\boldsymbol{A}$ and $\neg \boldsymbol{A}$ (there may be none). The desired cut-free tableau $\boldsymbol{X}$ shown on the right is like $\boldsymbol{Xp}$ but with every such branch extended by the tableau $\boldsymbol{Xn}$. This closes the branch instead of the removed formula $\boldsymbol{A}$ which cannot be used as a premise of any rule.

The case where $\boldsymbol{A}$ is a negation formula requires an inversion of negation rule, the cases where $\boldsymbol{A}$ is $\perp$ or $\top$ are straightforward and the remaining cases are variations of the presented ones. We leave them to the reader. ■

Theorem 26 (Cut elimination) If $\mathbf{T}$ and $\boldsymbol{A}$ are pure then $\mathbf{T} \vdash \boldsymbol{A}$ iff $\mathbf{T} \vdash^{+} \boldsymbol{A}$.

Proof: $(\rightarrow)$: obvious. $(\leftarrow)$: Let us assume that $\boldsymbol{X} : \mathbf{T} \vdash^{+} \boldsymbol{A}$. We first convert the tableau $\boldsymbol{X}$ into a regular atomically closed tableau $\boldsymbol{X}_1$ by the lemmas 23 and 16. Then we decrease (if needed) the cut rank $\boldsymbol{\rho}(\boldsymbol{X}_1)$ of $\boldsymbol{X}_1$ by one by recursive applications (starting from the bottom) of the reduction lemma 25 to the conclusions of all cuts of cut rank $\boldsymbol{\rho}(\boldsymbol{X}_1)$ occurring in $\boldsymbol{X}_1$.

This process is repeated until we get a tableau $\boldsymbol{X}_2$ of cut rank 0, i.e. $\boldsymbol{X}_2 : \mathbf{T} \vdash \boldsymbol{A}$. ■

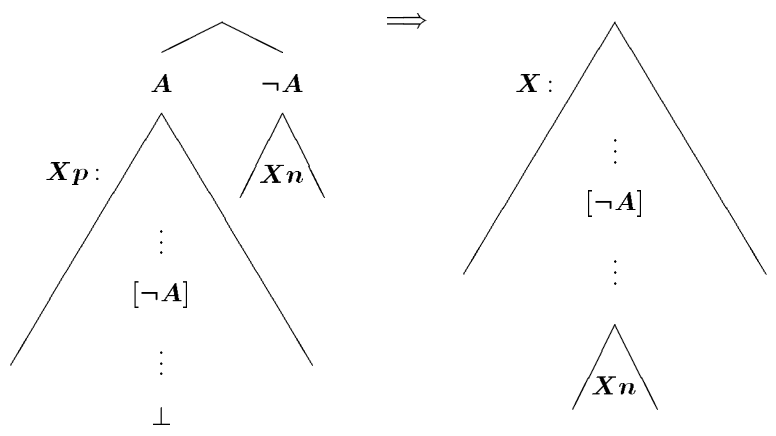


Figure 8: Reduction of an atomic cut formula

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